# Shnirelman peak in level spacing statistics of Calogero-like three-body problem

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Statistical properties of quantum quasidegeneracy in a Calogero-like three-body problem is presented. The hidden continuous symmetry of a Calogero problem is broken by adding a three-body interaction, which results in discrete symmetry. This symmetry is sufficient to get the Shnirelman peak in level spacing statistics. Our calculation immediately implicates the application of Shnirelman theorem in real physical quantum systems.

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## I. INTRODUCTION

Energy-level statistics is an important property of a quantum system, since it indicates the type of internal motion present in the system. This problem has drawn revival interest in different contexts. Now it is a well-established fact that distribution of nearest-neighbor level spacings is Poissonian for integrable systems, indicating no correlations between spacings. Whereas level repulsion, i.e., Wigner statistics is related with classically chaotic systems. In a classic paper [1], Bohigas conjectured that the Poissonian universality class of spectral fluctuations to be distinguished from GOE and GUE ensembles of random matrices depending on the presence of time-reversal symmetry (GOE) or not (GUE). Later Casati *et al.* [2], Seligman and Verbaarschot [3] provide some new discoveries in this direction. In 1993, Shnirelman showed that for systems with time reversal symmetry, one should observe a delta function peak of finite width at $s=0$ in the nearest-neighbor spacing distribution $p(s)$. It is known as the Shnirelman peak [4]. Apparently, it was in sharp contrast with the traditional level spacing statistics. Later it was understood that this peak appears due to the presence of symmetry. Separating levels by symmetry, one will get back Poisson distribution. Actually, the Shnirelman peak in the level spacing distribution indicates the presence of bulk quasidegeneracy, which has later been verified by Chirikov and Shepelyansky [5]. They have studied kicked rotator on a torus with time-reversal symmetry. The bulk quasidegenerate states are connected with time-reversal symmetry in their model.

But so far the application of the Shnirelman theorem is highly restricted to a specific case. In this communication, we present a more general model, a Calogero-like problem, which is widely used as a very useful model in different branches of physics. In 1969, in a nice paper [6], Calogero presented the complete solution of the Schrödinger equation for three particles interacting pairwise by two-body harmonic and inverse-square potential given by

$$V_c=\frac{1}{8}\omega^2\sum_{i<j}(x_i-x_j)^2+g\sum_{i<j}(x_i-x_j)^{-2}, \tag{1}$$

where $g>-\frac{1}{2}$ to avoid collapse of the system. Then it was extended to the exact solutions of a many-body problem (the Calogero-Sutherland model) [7,8]. Recently, it has attracted renewed interest in connection with spin chain problems [9,10]. The Hamiltonian of SU($n$) spins is given by

$$H=\sum_{j<k}h_{jk}P_{jk} \tag{2}$$

with inverse square exchange interaction

$$h_{jk}=1/[(x_j-x_k)^2], \tag{3}$$

where $P_{jk}$ is the operator that exchanges the spins at lattice sites. $x_j$ are the static equlibrium positions of particles in classical $N$-body Calogero systems [6,7]. In the Haldane-Shastry model [9,10], exact eigenstates of the family of $s=\frac{1}{2}$ antiferromagnetic Heisenberg chain with $1/r^2$ exchange interaction has been studied. The energy spectrum exhibits degenerate "super-multiplet" structure, which suggest hidden continuous symmetry present in the system. Very recently, it has been shown that this structure is the consequence of a Yangian symmetry [11], which is intimately related with integrability.

Due to the exact level degeneracy in the Calogero problem, the nearest-neighbor level spacing distribution will have the trivial shape of a delta function. This exact degeneracy comes from hidden continuous symmetry, which can be broken introducing a suitable perturbation to $V_c$. Now for a system of many interacting particles, it is difficult to determine the key mechanism that generates the statistical behavior. So here we restrict ourselves to $N=3$, where we can clearly visualize the underlying mechanism played by the mutual interaction as well. We lift the continuous symmetry of the Calogero three-body problem by adding a simple three-body interaction

$$V_{per}=\frac{\sqrt{3}f}{2r^2}\left[\frac{x_1+x_2-2x_3}{x_1-x_2}+\text{cyclic terms}\right], \tag{4}$$

where

$$r^2=\tfrac{1}{3}[(x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2]. \tag{5}$$

The reason to choose such a perturbing term: the problem is still integrable and algebraically solvable. One can calculate

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very high-energy levels quite accurately from a simple analytic expression. Although the continuous symmetry is broken by this perturbation, discrete symmetry still remains. It results to a bulk of quasidegeneracy leading to Shnirelman peak at $s=0$ in $p(s)$ distribution. Our calculation not only gives a physical interpretation of Shnirelman theorem but it immediately implicates its application to a class of quantum systems. Being a three-body system, we expect to get more pronounced Shnirelman effects here.

The paper is organized as follows. To make the paper self-contained in Sec. II, we present the concept of supersymmetric quantum mechanics and shape invariance, which is used to solve the Calogero problem analytically in the next section. Section III deals with the exact analytic solution of the chosen potential. Numerical results, discussion, and final conclusions are presented in Sec. IV.

## II. SUPERSYMMETRIC QUANTUM MECHANICS AND CONCEPT OF SHAPE INVARIANCE

Since the last few decades, it has been proved that supersymmetric quantum mechanics (SUSY-QM), together with the shape invariance condition, is the most compelling technique for exact solvability [12]. For a quantum-mechanical problem with a potential $V_1(x)$, supersymmetry allows one to construct a partner potential $V_2(x)$, which are isospectral, i.e., $E_{n+1}^{(1)}=E_n^{(2)}$. In SUSY-QM, one starts with the Schrödinger equation in the shifted energy scale, where $E_0^{(1)}=0$:

$$H_1\psi_0(x)=[-(d^2/dx^2)+V_1(x)]\psi_0=0. \tag{6}$$

Now defining superpotential $W(x)$ in terms of the ground-state wave function as

$$W(x)=-\psi_0'/\psi_0, \tag{7}$$

it is easy to write $H_1$ in terms of $W$ as

$$H_1=-(d^2/dx^2)+W^2-W'. \tag{8}$$

Then its supersymmetric partner becomes

$$H_2=-(d^2/dx^2)+W^2+W'. \tag{9}$$

$H_1$ and $H_2$ are called two-partner Hamiltonians, where the two-partner potentials are

$$\begin{aligned} V_1&=W^2-W', \\ V_2&=W^2+W'. \end{aligned} \tag{10}$$

Now the total SUSY Hamiltonian is given by

$$H=\{Q,Q^\dagger\}=\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \tag{11}$$

where $Q$, $Q^\dagger$, represent the supercharges, whose explicit forms are

$$Q=\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \quad \text{and} \quad Q^\dagger=\begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix}. \tag{12}$$

$A$ and $A^\dagger$ are called generalized annihilation and creation operators, respectively, and can be written in terms of superpotential as

$$A=(d/dx)+W(x), \quad A^\dagger=-(d/dx)+W(x). \tag{13}$$

Then, $H_{1,2}$ are factorizable as

$$H_1=A^\dagger A, \quad H_2=AA^\dagger. \tag{14}$$

The relations obeyed by $Q$, $Q^\dagger$, and $H$ satisfy the closed superalgebra $sl(1/1)$

$$[H,Q]=[H,Q^\dagger]=0 \tag{15}$$

and

$$\{Q^\dagger,Q^\dagger\}=\{Q,Q\}=0; \quad \{Q,Q^\dagger\}=H. \tag{16}$$

SUSY algebra can explicitly show the correspondence between $E_{n+1}^{(1)}$ and $E_n^{(2)}$. Let the eigenfunctions of $H_{1,2}$ that correspond to eigenvalues $E_n^{(1,2)}$ be $\psi_n^{(1,2)}$. One can easily see that for $n\neq 0$,

$$H_2(A\psi_n^{(1)})=AA^\dagger(A\psi_n^{(1)})=AH_1\psi_n^{(1)}=E_n^{(1)}(A\psi_n^{(1)}). \tag{17}$$

Thus, for $n>0$, $A\psi_n^{(1)}$ is an eigenfunction of $H_2$, which is a supersymmetric partner state of $\psi_n^{(1)}$. Since $A\psi_0^{(1)}=0$, the ground state of $V_1(x)$ does not have a SUSY partner and one finds $E_{n+1}^{(1)}=E_n^{(2)}$. Thus, SUSY algebra [12] shows that the pair of potentials $V_{1,2}$ have the same eigenspectrum, only the ground state of $V_1$ will be missing in $V_2$ (good supersymmetry). Now shape invariance means: if the pair of SUSY partners $V_{1,2}$ are similar in shape and differ only in parameter, i.e.,

$$V_2(x;a_1)=V_1(x;a_2)+R(a_1), \quad a_2=f(a_1), \tag{18}$$

then $V_{1,2}$ are said to be shape invariant. The shape invariance condition is an integrability condition. Using this condition in the hierarchy of Hamiltonians one can easily obtain the energy eigenvalues and eigenfunctions of any shape invariant potential analytically. The complete eigenspectrum of $H_1$ is then given by

$$E_n^{(1)}=\sum_{k=1}^{n} R(a_k), \quad E_0^{(1)}=0. \tag{19}$$

## III. EXACT ANALYTIC SOLUTION OF CALOGERO PROBLEM

Using the concept of supersymmetric quantum mechanics [12] and shape invariance condition, one can solve the full three-body problem $V=V_c+V_{per}$ in a simple algebraic manner (Sec. 4.5 of Ref. [12]). The full three-body potential of a modified Calogero problem has the following form:

$$V(x_1,x_2,x_3)=\tfrac{1}{8}\,\omega^2\sum_{i<j}(x_i-x_j)^2+g\sum_{i<j}(x_i-x_j)^{-2}+\frac{\sqrt{3}f}{2r^2}\left[\frac{x_1+x_2-2x_3}{x_1-x_2}+\text{cyclic terms}\right].\tag{20}$$

Following Calogero notation, one can map $V(x_1,x_2,x_3)$ to $V(r,\phi)$. Define the Jacobi coordinates as

$$R=\tfrac{1}{3}(x_1+x_2+x_3),$$
$$x=(x_1-x_2)/\sqrt{2},\tag{21}$$
$$y=(x_1+x_2-2x_3)/\sqrt{6}.$$

Eliminating the center-of-mass motion, the three-body problem can be reduced to effectively a one-body problem in two dimensions

$$x=r\sin\phi,\qquad y=r\cos\phi,\tag{22}$$

with $r$ and $\phi$ range; $0\leqslant r\leqslant\infty$ and $0\leqslant\phi\leqslant 2\pi$.

Then, $V(x_1,x_2,x_3)$ can be easily tranformed to polar coordinates as

$$V(r,\phi)=\frac{3}{8}\omega^2r^2+\frac{1}{r^2}\left[\frac{9}{2}g\csc^2 3\phi+\frac{9}{2}f\cot 3\phi\right]=V(r)+[V(\phi)/r^2]\tag{23}$$

using the identities

$$\sum_{m=1}^{3}\csc^2[\phi+2(m-1)(\pi/3)]=9\csc^2 3\phi,$$
$$\sum_{m=1}^{3}\cot[\phi+2(m-1)(\pi/3)]=3\cot 3\phi.\tag{24}$$

Note that $V(r,\phi)$ is separable in the $(r,\phi)$ coordinate and the total wave function can be written as

$$\psi_{nl}(r,\phi)=\frac{R_{nl}(r)}{\sqrt{r}}F_l(\phi).\tag{25}$$

Supersymmetric quantum mechanics shows that $V(r,\phi)$ is shape invariant in $r$ and $\phi$ coordinates separately and one can get the full energy spectrum algebraically.

The radial Schrödinger equation

$$\left[-\frac{d^2}{dr^2}+\frac{3}{8}\omega^2r^2+\frac{\left(B_l^2-\frac{1}{4}\right)}{r^2}\right]R_{nl}=E_{nl}R_{nl}(r)\tag{26}$$

corresponds to shape invariant potential with superpotential

$$W(r)=\sqrt{\frac{3}{8}}\omega r-\frac{B_l+\frac{1}{2}}{r}.\tag{27}$$

$B_l^2$ is the energy eigenvalue of the Schrödinger equation in the angular variable. Since $V(r)=\frac{3}{8}\omega^2r^2+(B_l^2-\frac{1}{4})/r^2$ is shape invariant with $W(r)$ given by Eq. (27), the radial Schrödinger equation is algebraically solvable and the energy eigenvalues are obtained in a closed form

$$E_{nl}=\sqrt{\tfrac{3}{2}}\,\omega(2n+B_l+1),\quad n=0,1,2,\ldots,\quad l=0,1,2\ldots.\tag{28}$$

The Schrödinger equation in angular variable is

$$\left[-\frac{d^2}{d\phi^2}+\frac{9}{2}g\csc^2 3\phi+\frac{9}{2}f\cot 3\phi\right]F_l(\phi)=B_l^2F_l(\phi),\tag{29}$$

which again corresponds to shape invariant potential in angular variable with superpotential

$$W(\phi)=-3(a+1/2)\cot 3\phi-\frac{3}{4}\frac{f}{a+1/2},$$
$$a=1/2(1+2g)^{1/2}.\tag{30}$$

It results in an analytic expression of $B_l^2$:

$$B_l^2=9(l+a+1/2)^2-\frac{9}{16}\frac{f^2}{(l+a+1/2)^2}.\tag{31}$$

It is easy to check that with $f=0$ limit one can recover the results for the Calogero potential $V_c$, which results in highly degenerate multiplets.

## IV. RESULTS AND DISCUSSION

We calculate the lowest 10 000 energy levels from Eq. (28) using Eq. (31) with very high precision. In our double precision calculation we keep 15 valid digits. Before calculating nearest-neighbor level spacing distribution $p(s)$, we first unfold the spectrum, just to get rid of $N_{av}(E)$. Let $\{E_i\}$ be a sequence of discrete spectrum and $N(E)$ is the spectral staircase function that counts the number of levels below $E$. Now it is possible to separate $N(E)$ in a smooth part $N_{av}(E)$ and a fluctuating part $N_{fl}(E)$:

$$N(E)=N_{av}(E)+N_{fl}(E).\tag{32}$$

Unfolding is done through some mapping $E\rightarrow\epsilon$,

$$\{\epsilon_i\}=\{N_{av}(E_i)\},\quad i=1,2,\cdots.\tag{33}$$

The sequence $\{\epsilon_i\}$ now has unit mean spacing. The level spacing is $s_i=\epsilon_{i+1}-\epsilon_i$. Now $p(s)$ distribution will apply to this sequence $\{\epsilon_i\}$.

For unfolding, we use cubic spline smoothing and calculate $p(s)$ distribution from the unfolded spectrum. Our re-

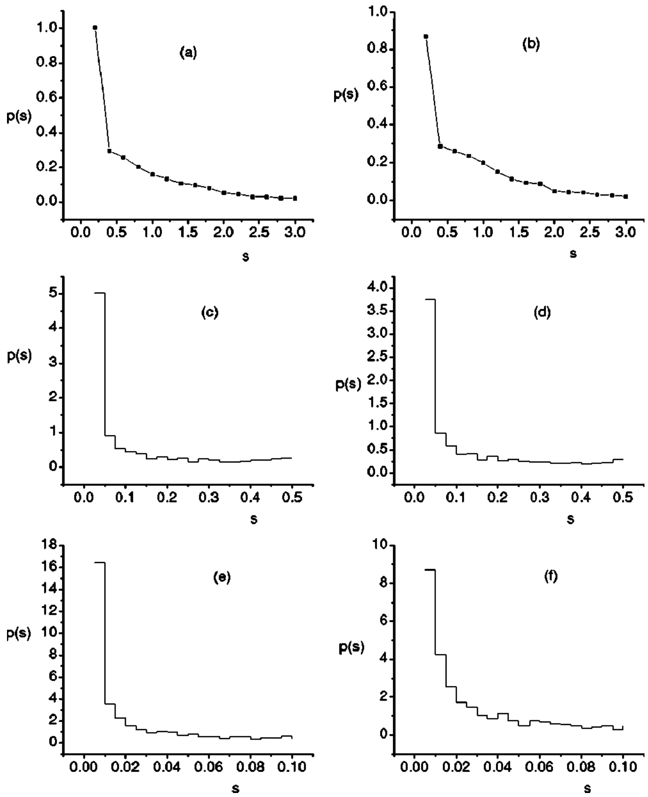


FIG. 1. Level spacing distribution $p(s)$ with $g=-0.1$, $f=2.0$ (left column) and $g=1.0$, $f=5.0$ (right column).

sults are presented in Figs. 1 and 2. In Fig. 1, we plot nearest-neighbor level spacing distribution $p(s)$. In Fig. 2, we present integral level spacing distribution $I(s)=Np(s)$, where the spacing integral probability is normalized to unity. As a representative calculation we take the parameters $g=-0.1$, $f=2.0$, and $g=1.0$, $f=5.0$. The left column of Fig. 1 corresponds to the first set of parameters and the right column of Fig. 1 corresponds to the second set of parameters. In both cases, the large peak appears in the first bin of the histogram. To see the distribution in finer details near $s=0$, we plot them in Figs. 1(c)–1(f), where the Shnirelman peak in the first bin clearly demonstrates the existence of global quasidegeneracy. It unambiguously supports the old theorem predicted by Shnirelman. The resolution of the peak

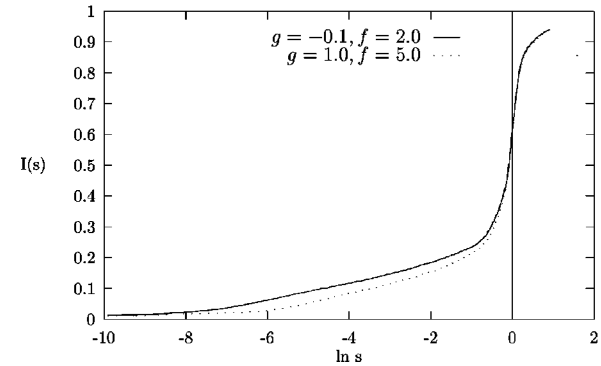


FIG. 2. Normalized integral level spacing distribution with the same parameters as Fig. 1.

is presented in Fig. 2. It has two different regions. The leftmost part is most interesting. It shows the linear dependence of $I$ on $ln\ s$, which represents the structure of the Shnirelman peak, whereas the rightmost steep increase part represents the Poissonian tail. The results with much higher three-body interaction (large value of $f$) are not presented here. We have checked that a higher value of $f$ cannot lift the effect of quasidegeneracy completely. So for this integrable perturbation the effect of global quasidegeneracy remains.

In conclusion, we want to mention that our results present the appearance of the Shnirelman peak in the level spacing distribution of a very important integrable model, which is widely used for realistic physical problems in different branches of physics. Our results clearly prove that discrete symmetry present in the quantum system is sufficient for the appearance of the Shnirelman peak. Being a three-body model, the Shnirelman effect is much pronounced here. Our calculation nicely demonstrates how a three-body interaction can be used as perturbation to lift hidden continuous symmetry when discrete symmetry still remains. Being an analytically solvable model, its extension to $n$-body problem is quite straightforward where one can expect much rich structure of quasidegeneracy.

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